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Propagation of water waves past long two-dimensional obstacles

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An approximate analysis is developed for the propagation of water waves past long obstacles by considering separately the effects of diffraction at each end. The motion is two-dimensional, and linearized potential flow is assumed. Reflexion and transmission coefficients are obtained for the long obstacle, and it is shown that for suitably chosen values of the obstacle length there is complete transmission due to interference between the two ends. A comparison is made with experiments for the case of a rectangular obstacle.

1. Introduction

Diffraction occurs when incoming progressive waves on the surface of a heavy fluid are incident upon a floating or submerged obstacle. If the obstacle is a long horizontal cylinder parallel to the incoming wave crests, the resulting fluid motion will normally be two-dimensional, i.e. confined to planes perpendicular to the cylinder's generators. In this case the incident wave will be partially transmitted past the obstacle, and partially reflected back. If the incident wave is harmonic in time it is convenient to describe the resulting motion in terms of transmission and reflexion coefficients; these are defined as the complex amplitudes of the transmitted and reflected waves at infinity, divided by the incident wave amplitude.

There have been numerous investigations of the problem described above, primarily for specific simple cylinders. Bibliographies may be found in the survey articles of Wehausen & Laitone (1960) and Wehausen (1963). A very general analysis of obstacle problems was presented by Kreisel (1949), including symmetry relations between the reflexion coefficients for waves incident from either direction.

The present paper is concerned with the approximate analysis of two-dimensional wave reflexion and transmission in the special case where the obstacle is symmetrical and very long in the horizontal co-ordinate, compared to the wavelength, and where there is a long horizontal middle portion of the obstacle. With this geometrical configuration it is natural to consider the obstacle as made up of two shorter obstacles, each corresponding to one end of the long one, and joined by a long region of constant depth. One may then envisage the incident wave to be partially reflected at the first obstacle and partially transmitted to the second, where a smaller part is transmitted on and another part reflected back to the first; this sequence of reflexion and transmission at each end can be continued as a geometric progression, converging to the solution for the long obstacle and including one reflected wave, one transmitted wave, and progressive waves propagating in each direction between the two ends. The necessary analysis is easier to perform than describe. Before doing so, however, we shall review and extend the derivation of Kreisel's symmetry proof for a single obstacle, since it can be used to draw important conclusions concerning the reflexion and transmission properties of the long obstacle. In particular we shall show that there exists an infinite set of wavelengths for a long symmetrical obstacle, such that the incident wave is totally transmitted. As a corollary it follows that the reflexion coefficient is a highly oscillatory function of the obstacle length, when this is large compared to the wavelength.

A similar treatment of obstacles is that of Biesel & Le Méhauté (1955) but that work is limited to obstacles composed of a pair of short, individually symmetrical obstacles. This precludes the analysis, for example, of obstacles with a long elevated horizontal surface such as the submerged rectangular parallelepiped which is treated by Jolas (1960) and Takano (1960).

2. The boundary-value problem and symmetry relations

Let (x, y) be Cartesian co-ordinates, with $y = 0$ representing the plane of the undisturbed free surface and y being positive downwards. The fluid occupies the region $-\infty < x < \infty$, $0 < y < h(x)$, with the possible exception of one or more

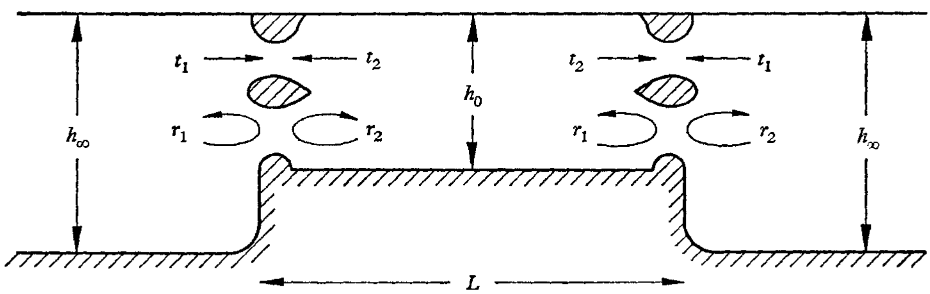

FIGURE 1. The geometrical configuration of the symmetrical obstacle.

obstacles which are situated in the fluid within a finite distance of the origin (figure 1). The function $h(x)$ is the fluid depth and we assume that $h(x) > 0$ for $-\infty < x < \infty$ and that $h(x)$ tends to two distinct finite or infinite limits, say $h^{\pm}$, as $x \to \pm\infty$. With this geometrical configuration, diffraction of water waves will occur due to the variation of depth or to the obstacle(s) situated in the fluid, or to both.

We consider the propagation of small plane progressive waves, sinusoidal in time with frequency $\sigma/2\pi$. If one assumes irrotational incompressible flow, the fluid-velocity vector may be represented by

$$\mathbf{v} = \operatorname{Re}\,[e^{-i\sigma t}\nabla\phi(x, y)], \tag{2.1}$$

where $\phi(x, y)$ is the complex velocity potential. This potential must satisfy Laplace's equation in the fluid domain, the kinematic boundary condition

$$\partial\phi/\partial n = 0 \tag{2.2}$$

on the bottom and obstacles, and the linearized free-surface condition

$$\sigma^2\phi + g\,\partial\phi/\partial y = 0 \quad \text{on} \quad y = 0, \tag{2.3}$$

with g the gravitational acceleration.

In the general case we allow incident waves from both directions, and at infinity the potential will be of the asymptotic form

$$\phi \to (A^{\pm} e^{-iK^{\pm}x} + B^{\pm} e^{iK^{\pm}x}) \cosh K^{\pm}(y - h^{\pm}) \quad \text{as} \quad x \to \pm\infty. \tag{2.4}$$

Here $A^{\pm}$ and $B^{\pm}$ are complex constants, with A^{+} and B^{-} representing the incoming waves from right and left, respectively, and B^{+} and A^{-} the outgoing waves. The wave-numbers $K^{\pm}$ are defined as the real positive roots of the equations

$$K^{\pm} \tanh K^{\pm}h^{\pm} = \sigma^2/g. \tag{2.5}$$

Following Kreisel (1949), we denote the potential symbolically by

$$\phi = \{A^{+}, B^{+};\ A^{-}, B^{-}\}. \tag{2.6}$$

Usually the physical problem is that of incident waves only from one direction, i.e. either $B^{-} = 0$, for waves incident only from $x = +\infty$, or $A^{+} = 0$, for waves incident only from $x = -\infty$. We denote the corresponding potentials by

$$\phi_1 = \{A_1^{+}, B_1^{+};\ A_1^{-}, 0\} \tag{2.7}$$

and

$$\phi_2 = \{0, B_2^{+};\ A_2^{-}, B_2^{-}\}. \tag{2.8}$$

Now, if ϕ_1 is a solution for the prescribed boundary conditions, then so is its complex conjugate

$$\overline{\phi_1} = \{\overline{B_1^{+}}, \overline{A_1^{+}};\ 0, \overline{A_1^{-}}\},$$

and likewise any linear combination of ϕ_1 and $\overline{\phi_1}$ will be a solution. In particular,

$$\phi_2 = A_1^{+}\overline{\phi_1} - \overline{B_1^{+}}\phi_1 = \{0, |A_1^{+}|^2 - |B_1^{+}|^2;\ -\overline{B_1^{+}}A_1^{-}, A_1^{+}\overline{A_1^{-}}\} \tag{2.9}$$

is a possible solution of the second problem, (2.8), in terms of the first, (2.7).

The elevation of the free surface is given by the linearized formula

$$\eta = \frac{1}{g}\frac{\partial[\phi(x,0)\,e^{-i\sigma t}]}{\partial t} = -\frac{i\sigma}{g}\phi(x,0)\,e^{-i\sigma t}. \tag{2.10}$$

Thus the reflexion coefficients for the two problems are

$$R_1 = B_1^{+}/A_1^{+}, \tag{2.11}$$

$$R_2 = A_2^{-}/B_2^{-}, \tag{2.12}$$

and the corresponding transmission coefficients are

$$T_1 = (A_1^{-}\cosh K^{-}h^{-})/(A_1^{+}\cosh K^{+}h^{+}), \tag{2.13}$$

$$T_2 = (B_2^{+}\cosh K^{+}h^{+})/(B_2^{-}\cosh K^{-}h^{-}). \tag{2.14}$$

We note that these coefficients are complex, representing both the magnitude and phase of the diffraction process.

Using equation (2.9) to replace the coefficients $A_2^{\pm}$ and $B_2^{\pm}$, it follows from (2.12) and (2.14) that

$$R_2 = -\overline{B_1^+}\, A_1^- / A_1^+ \overline{A_1^-}, \tag{2.15}$$

$$T_2 = \frac{|A_1^+|^2 - |B_1^+|^2}{A_1^+\overline{A_1^-}} \frac{\cosh K^+h^+}{\cosh K^-h^-}. \tag{2.16}$$

Comparing (2.11) and (2.15), we get the first symmetry result

$$|R_1| = |R_2| \equiv R, \tag{2.17}$$

stating that for any obstacle the amplitude of the reflexion coefficient is independent of the direction of the incident wave. This fundamental symmetry relation is due to Kreisel (1949). Forming the product of T_1 and T_2 from (2.13) and (2.16), we obtain the additional relation

$$|T_1 T_2| = 1 - |R|^2. \tag{2.18}$$

Similar formulae can be inferred regarding the phases. Thus it follows from (2.13) and (2.16) that

$$\arg T_1 = \arg T_2 \equiv \delta T, \tag{2.19}$$

and from (2.11) and (2.15) that

$$\arg R_1 + \arg R_2 = \pi + 2\delta T. \tag{2.20}$$

3. Long symmetrical obstacles

We are now ready to analyse the problem formulated in §1. The obstacle is long compared to a wavelength and can be composed of two identical short obstacles, placed back-to-back and joined by a long region of constant depth. The short obstacles are situated at $x = 0$ and $x = L$. The fluid depth is h_∞ at $x = \pm\infty$, and h_0 in the region between the two short obstacles. No restrictions are placed on the relative magnitudes of h_0 and h_∞.

Incident waves of unit amplitude are assumed to propagate from $x = +\infty$, with no incoming waves from $x = -\infty$. As a result there will be outgoing reflected waves, of amplitude R, at $x = +\infty$, and outgoing transmitted waves, of amplitude T, at $x = -\infty$.

Let K_∞ and K_0 denote the wave-numbers appropriate to the depths h_∞ and h_0, as obtained from equation (2.6). If $K_0 L \gg 1$ the wave motion in the middle portion of the long obstacle will approximate to two plane waves, moving in opposite directions. Thus we can assume the following asymptotic behaviour for the wave height:

$$\eta \sim (e^{-iK_\infty x} + R\, e^{iK_\infty x})\, e^{-i\sigma t} \quad (x \to +\infty),$$

$$\eta \sim (A\, e^{-iK_0 x} + B e^{iK_0 x})\, e^{-i\sigma t} \quad (1 \ll K_0 x \ll K_0 L),$$

$$\eta \sim T e^{-iK_\infty x - i\sigma t} \quad (x \to -\infty).$$

The four coefficients R, T, A, and B are unknown, but they can be found from four simultaneous equations obtained by matching the waves at each short obstacle according to the reflexion and transmission coefficients for the short obstacles. These are assumed known, and are denoted by lower-case letters as indicated in figure 1. Matching at $x = L$ then gives the two equations

$$A\,e^{-iK_0 L} = t_1 e^{-iK_\infty L} + Br_2 e^{iK_0 L}, \tag{3.1}$$

$$Re^{iK_\infty L} = r_1 e^{-iK_\infty L} + Bt_2 e^{iK_0 L}, \tag{3.2}$$

and matching at $x = 0$ gives

$$B = r_2 A, \tag{3.3}$$

$$T = t_2 A. \tag{3.4}$$

Solving for the four unknowns, we obtain

$$A = t_1 e^{i(K_0 - K_\infty)L}/[1 - r_2^2 e^{2iK_0 L}], \tag{3.5}$$

$$B = r_2 t_1 e^{i(K_0 - K_\infty)L}/[1 - r_2^2 e^{2iK_0 L}], \tag{3.6}$$

$$T = t_1 t_2 e^{i(K_0 - K_\infty)L}/[1 - r_2^2 e^{2iK_0 L}], \tag{3.7}$$

$$R = r_1 e^{-2iK_\infty L} + r_2 t_1 t_2 e^{2i(K_0 - K_\infty)L}/[1 - r_2^2 e^{2iK_0 L}]. \tag{3.8}$$

Since, from (2.18), $|t_1 t_2| = 1 - |r_2|^2$, the magnitude of the transmission coefficient is

$$|T| = [1 - |r_2|^2]/[|1 - |r_2|^2 e^{2iK_0 L + 2i\delta r_2}|], \tag{3.9}$$

where δr_2 is the phase of the reflexion coefficient r_2. It follows immediately that if the length L is suitably chosen to satisfy

$$K_0 L + \delta r_2 = n\pi, \tag{3.10}$$

for any integer n, then $|T| = 1$ and the long obstacle is 'transparent'; i.e. the incident wave is totally transmitted without reflexion. Clearly in this case $R = 0$, as can be verified directly by using the phase relation (2.20). We see therefore that the reflexion coefficient R will depend critically on the parameter $K_0 L$, or the ratio of obstacle length to wavelength. Since K_0 depends in turn on the depth h_0, a long obstacle will have a reflexion coefficient which depends critically on both its length and its depth.

As long as $|r| < 1$, the wave motion is bounded and there is no resonance effect in the middle of the long obstacle. If we allow $|r| \to 1$, implying that the short obstacles are almost perfectly reflecting,† then a singular behaviour follows when (3.10) is satisfied, since the denominator of the expressions in (3.5)–(3.9) tend to zero. However, it is imperative to note that, unless (3.10) is satisfied exactly, two limit processes are involved, and the final result depends on the order in which they are obtained. If $|r| = 1 - \epsilon_1$ and $K_0 L + \delta r_2 = n\pi + \epsilon_2$ with ϵ_1 and ϵ_2 both small, then

$$|T| \simeq (1 + \epsilon_2^2/\epsilon_1^2)^{-\frac{1}{2}}.$$

Thus the transmission coefficient can take any value between zero and one, depending on the ratio ϵ_2/ϵ_1. If this ratio is small, implying that

$$|K_0 L + \delta r_2 - n\pi| \ll 1 - |r| \ll 1,$$

then in the limit $T = 1$ and $R = 0$, while A and B are infinite. In this case there is zero reflexion and unbounded resonance on the obstacle, as was described by Biesel & Le Méhauté (1955). However if the ratio ϵ_2/ϵ_1 is large, implying that

$$1 - |r| \ll |K_0 L + \delta r_2 - n\pi| \ll 1,$$

† Obvious examples are a vertical barrier with a narrow slit at or beneath the surface, or a barrier extending from the free surface down to a large depth.

then the order of the two limits is reversed and it follows instead that $T = 0$ and $|R| = 1$. This is the situation one would normally envisage of complete reflexion from the first barrier and no transmission beyond the second. Nevertheless, the coefficients A and B, describing the wave amplitude on the obstacle, are still unspecified unless one prescribes the behaviour of t_1 as $|r| \to 1$. Thus it is clear that the limit $|r| \to 1$ is quite complicated, and one must be very careful to specify the precise situation to be represented.

Finally, we note that the analysis of the long obstacle can be generalized to non-symmetric profiles, by allowing the reflexion and transmission coefficients of the two ends to differ. Equations comparable to (3.5)–(3.9) are readily obtained. For example, the amplitude of the transmission coefficient is given by

$$|T| = |t_1 t_2'|/[|1 - r_2 r_2' e^{2iK_0L}|], \tag{3.11}$$

where the primed coefficients correspond to the downstream end and the unprimed to the upstream end. Complete transmission is possible if and only if $|r| = |r'|$.

4. Conclusions

We have derived the reflexion and transmission coefficients of a long horizontal obstacle in terms of the coefficients for each end. The results are asymptotically correct assuming the obstacle to be long compared to a wavelength. The reflexion and transmission coefficients depend critically on the parameter $K_0 L$, where K_0 is the wave-number for plane waves propagating above the mid-portion of the obstacle and L is its length. Thus there is a strong dependence on both the length and depth of the obstacle, which can be identified physically with interference between the waves reflected and transmitted at each end. These conclusions are independent of the detailed shape of the obstacle at each end.

Within the accuracy of the asymptotic approximation there is an infinite sequence of values of $K_0 L$ or, for any given obstacle geometry, of incident wavelengths, such that the obstacle is 'transparent'; i.e. the wave is totally transmitted with no reflexion taking place. Generally, however, there will be a phase shift. An analogous situation exists for submerged circular obstacles in infinitely deep water, where, as first shown by Dean (1948), there is complete transmission for *all* frequencies.

As an example our analysis can be applied to the case of a long rectangular obstacle, for which comparison may be made with the experiments of Takano (1960). We assume the depth to be infinite, except at the obstacle, so that the reflexion coefficient for the infinite step (Newman 1965) can be applied at each end. The results are shown in figure 2 for an obstacle of length $8{\cdot}86h_0$, corresponding to the experimental arrangement of Takano. In an exact treatment one could expect that the sharp zeros in the reflexion coefficient would be rounded off at non-zero minima, especially for the low values of $K_\infty h_0$. Nevertheless, the trend of the experimental points is in agreement with the computed curves, and it is clear that, due to interference phenomena, a complete description of this problem must include a much wider range of frequencies than was the case in Takano's experiments or the parallel theoretical treatment by Jolas (1960).

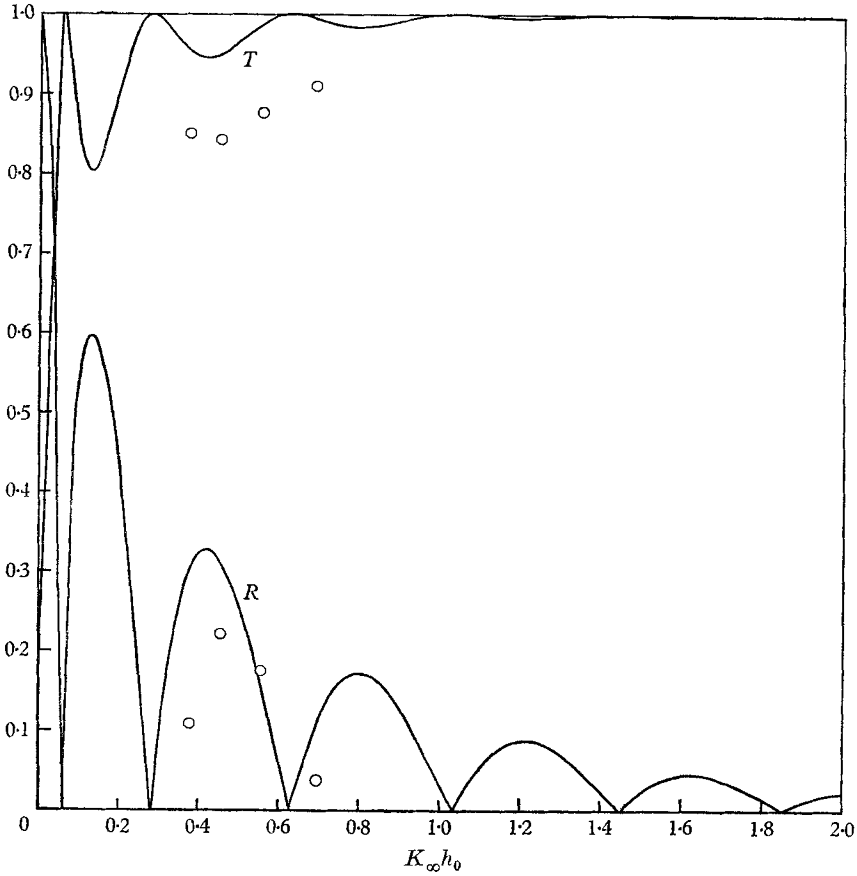

FIGURE 2. Approximate reflexion and transmission coefficients for the rectangular parallelepiped of length $8{\cdot}86h_0$ in infinitely deep water. Experimental points of Takano (1960) are shown.

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